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1997 J. Phys.: Condens. Matter 9 1123

(http://iopscience.iop.org/0953-8984/9/5/016)

J. Phys.: Condens. Matter **9** (1997) 1123–1132. Printed in the UK PII: S0953-8984(97)77847-0

The re-entrant phase diagram of the generalized Harper equation

Jukka A Ketoja† and Indubala I Satija‡§

† Department of Mathematics, University of Helsinki, PO Box 4, FIN-00014 Helsinki, Finland
‡ Department of Physics and Institute of Computational Sciences and Informatics, George Mason University, Fairfax, VA 22030, USA

Received 16 September 1996, in final form 18 November 1996

Abstract. We study the phase diagram of the tight-binding model for an electron on an anisotropic square lattice with a four-dimensional parameter space defined by two nearest-neighbour and two next-nearest-neighbour couplings. Using a renormalization scheme, we show that the inequality of the two next-nearest-neighbour couplings destroys the fat critical regime found in the isotropic case above the bicritical line and replaces it with another re-entrant extended phase. The scaling properties of the model are those of the corresponding tight-binding models on the nearest-neighbour square and triangular lattices. The triangular universality class also describes the quantum Ising chain in a transverse field with the only exception being the conformally invariant state of the Ising model which has no analogue in the triangular-lattice case.

1. Introduction

The Harper equation [1] (also known as the almost Mathieu equation [2])

$$t_a(\psi_{k+1} + \psi_{k-1}) + 2t_b \cos[2\pi(k\sigma + \phi)]\psi_k = E\psi_k \qquad (1)$$

describes, in the tight-binding approximation, the two-dimensional nearest-neighbour (NN) electron gas on the square lattice in a transverse magnetic field. Here, t_a and t_b are the NN couplings along the x- and y-directions. This one-dimensional representation of the two-dimensional problem can be obtained for an electron in a strong (weak) two-dimensional periodic potential and weak (strong) magnetic field. The parameter σ, which is equal to the magnetic flux per plaquette in units of the flux quantum, plays an important role in this problem: if σ is irrational, the system is quasiperiodic. In particular, for diophantine σ the model exhibits a metal–insulator transition along one of the directions of the two-dimensional lattice [2]. For the anisotropic square lattice, $t_a < t_b$, states are extended (E) along the y-direction and localized (L) along the x-direction, with the inverse localization length $\gamma = \ln(t_b/t_a)$. In the isotropic limit $t_a = t_b$, the localization length is infinite in both directions, and the electron wave functions and the spectrum are multi-fractals. This exotic behaviour is in between the E- and L-type behaviours and has been called critical (C). For σ equal to the inverse golden mean, the quantum states as well as the spectrum exhibit self-similarity. Our recent studies on the Harper equation [3] have shown that such complexity and richness of the eigenstates is not a property of the C phase only but can be

§ E-mail: isatija@sitar.gmu.edu.

found also in the L phase if one studies the fluctuations about the exponentially localized states. The fractal characteristics of these states are described by a unique strong-coupling renormalization fixed point. This fixed point is distinct from the critical fixed point of the Harper equation. The existence of these fixed points implies universality in the sense that bounded perturbations on the periodic potential are also described by the universality class of the Harper equation.

In this paper, we study the generalization which results from taking into account both the NN and next-nearest-neighbour (NNN) interaction in the electron problem. The associated tight-binding equation (TBE) has the form [4]

$$\begin{aligned} t_a(\psi_{k+1}+\psi_{k-1}) &+ 2t_b\cos[2\pi(k\sigma+\phi)]\psi_k \\ &+ \exp[\mathrm{i}\,2\pi(k\sigma+\phi)]\{t_{ab}\mathrm{e}^{\mathrm{i}\pi\sigma}\psi_{k+1}+t_{a\bar{b}}\mathrm{e}^{-\mathrm{i}\pi\sigma}\psi_{k-1}\} \\ &+ \exp[-\mathrm{i}\,2\pi(k\sigma+\phi)]\{t_{ab}\mathrm{e}^{\mathrm{i}\pi\sigma}\psi_{k-1}+t_{a\bar{b}}\mathrm{e}^{-\mathrm{i}\pi\sigma}\psi_{k+1}\} = E\psi_k \end{aligned} \quad (2)$$

where t_{ab} and $t_{a\bar{b}}$ are the diagonal NNN couplings. The Harper equation corresponds to vanishing of the NNN couplings. Furthermore, in the limit $t_{a\bar{b}} = 0$, the model describes an electron on the NN triangular lattice.

We would like to emphasize that in this paper we mainly treat equation (2) as a one-dimensional generalization of the Harper equation which provides us with the possibility of testing the predictions of the Harper equation in a more general context. In particular, the phase diagrams that we obtain are for the one-dimensional representation of the problem. However, the topology of the two-dimensional lattice is reflected in transformations that we find between eigenfunctions in various regions of the phase diagram.

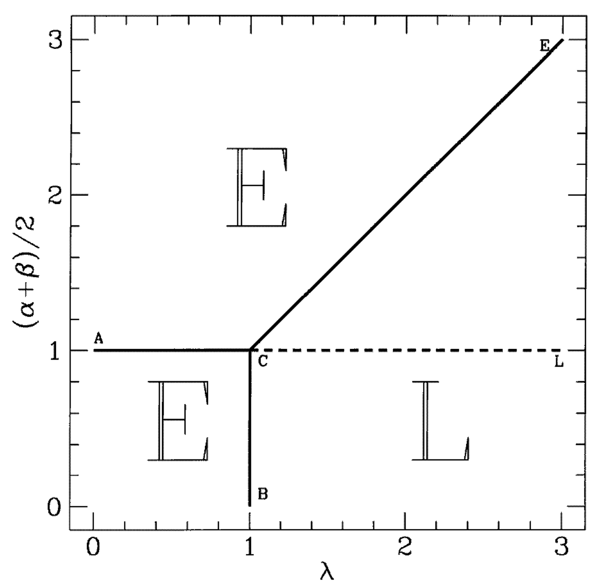

Figure 1. The phase diagram of equation (2). For $\alpha = \beta$, the region ACE is a critical phase which includes the lines AC and CE. For $\alpha \neq \beta$, ACE is extended and the critical behaviour exists only along the lines AC, BC, and CE. Furthermore, the scaling of the fluctuations along the line CL is different from those of the L regions above and below this line.

Recently, the above model was studied in the isotropic limit where $t_{ab} = t_{a\bar{b}}$ [4, 5, 3]. For $t_a > 2t_{ab}$, the phase diagram was described by the universality class of Harper equation.

However for $t_a \leqslant 2t_{ab}$, the behaviour was a lot more complex. One of the novel aspects of the two-parameter phase diagram (see figure 1) was the existence of a fat critical phase. Our renormalization scheme showed that the scaling properties of this critical phase were described by a universal strange attractor of the renormalization group (RG) equations.

The general case of the model where t_{ab} and $t_{a\bar{b}}$ are not equal and the resulting TBE is complex has not been fully investigated. Using the duality property of the model, Han *et al* (see [4]) calculated the Lyapunov exponent of the model analytically in the space of three parameters $\lambda = t_b/t_a$, $\alpha = 2t_{ab}/t_a$ and $\beta = 2t_{a\bar{b}}/t_a$. They concluded that the system is localized for $\lambda > 1$ if $(\alpha + \beta)/2 < 1$ and for $\lambda > (\alpha + \beta)/2$ otherwise. Apart from the existence of the localized phase, nothing has been known about the scaling properties of the complex model. Most importantly, no previous studies have explained how the fat critical phase above the bicritical line is affected by taking the two NNN couplings not equal.

In this paper, we study the complex model using our recently developed decimation scheme. We will confine ourselves to the case where $\sigma = (\sqrt{5} - 1)/2$. It is shown that the phase diagram changes discontinuously as $\alpha - \beta$ becomes different from zero. Furthermore, β turns out to be an irrelevant parameter (for $\alpha > \beta$) and hence the phase diagram can be determined by setting $t_{a\bar{b}}$ equal to zero. The interesting fat C phase of the case with equal NNN couplings is destroyed and the C–L transition is replaced by the E–L transition. In order to observe critical features, at least two out of the remaining three couplings t_a, t_b, and t_{ab} have to be equal and the third one cannot be bigger than the other two. Our detailed renormalization analysis shows that the universal features of the phase diagram are characterized by the universality classes of the NN square lattice (the Harper equation) or the triangular lattice. We also show that the triangular universality class describes the quantum Ising spin chain in a quasiperiodic, transverse magnetic field.

In section 2, we review our decimation scheme as applied to the NN TBEs. Section 3 summarizes our previous results in the case where the two NNN couplings are equal, and in section 4 we give the new results on the phase diagram of the anisotropic complex model. In section 5, we study the relationship of this problem with the quantum Ising chain in a quasiperiodic transverse field. We summarize our results in section 6.

2. The decimation scheme

We will use a decimation approach to describe the scaling properties of the wave function in the E and C phases and the fluctuations of the wave functions in the L phase, for a specific value of energy. In our studies below, we will focus on the quantum state with minimum energy E_{min}. In the above TBE with no mobility edges, all of the quantum states exhibit qualitatively the same features—that is, they are either E, L or C. They however do differ in the details of the scaling properties.

In addition to fixing the quantum state, one has to also fix the phase factor ϕ to a critical value in equation (2) so that the wave function remains finite asymptotically. The nondivergent wave functions are needed to determine the scaling properties as has been discussed previously [6, 7, 5].

The key idea of the decimation scheme is to connect the wave function ψ_k at an arbitrary site k with two neighbouring Fibonacci sites $k + F_{n+1}$ and $k + F_n$ where $F_{n+1} = F_n + F_{n-1}$ [7, 8]:

$$f_n(k)\psi(k + F_{n+1}) = \psi(k + F_n) + e_n(k)\psi(k). \tag{3}$$

The way in which the decimation functions f_n and e_n are placed in the decimation equation is somewhat arbitrary but here we choose the form which causes the asymptotic limits of the decimation functions e_n and f_n as $n \to \infty$ to be bounded in all three phases. The additive property of the Fibonacci numbers provides exact recursion relations for the decimation functions e_n and f_n [3, 7, 8]:

$$e_{n+1}(k) = -\frac{Ae_n(k)}{1 + Af_n(k)} \tag{4}$$

$$f_{n+1}(k) = \frac{f_{n-1}(k + F_n) f_n(k + F_n)}{1 + Af_n(k)} \tag{5}$$

$$A = e_{n-1}(k + F_n) + f_{n-1}(k + F_n)e_n(k + F_n).$$

For fixed k, the above coupled equations for the decimation functions define a RG flow which asymptotically ($n \to \infty$) converges on an attractor. The C phase is distinguished from the E phase by the existence of nontrivial limiting behaviour. With anisotropic NNN coupling, the attractor is a p-cycle in all three phases for $E = E_{min}$ [9]. The asymptotic p-cycle for $e_n(0)$ and $f_n(0)$ determines the universal scaling ratios

$$\zeta_j = \lim_{n\to\infty} \psi(F_{pn+j})/\psi(0) \qquad j = 0, \ldots, p - 1. \tag{6}$$

whose absolute values are equal to unity in the E phase and less than unity in the C phase [9].

In order to study the scaling properties of the L phase [3], we write

$$\psi_k = \mathrm{e}^{-\gamma|k|}\eta_k \tag{7}$$

where γ is the Lyapunov exponent which vanishes in the E and C phases and is positive in the L phase. Knowing the analytic formula for the Lyapunov exponent, it is easy to write a TBE for η_k, resembling equation (2). The decimation can be carried out for this modified TBE in the same way as for the original equation. In particular, there will be a scaling ratio ζ characterizing the fluctuations in the exponentially decaying wave function.

3. Isotropic NNN couplings: a review of the previous work

We briefly review the analysis of the TBE with NNN couplings in the isotropic limit where $t_{ab} = t_{a\bar{b}}$ [4, 5, 3]. Although the wave function ψ_k in general is a complex function of the lattice index k, ψ_k can be taken to be real in this case. For $2t_{ab} < t_a$, the model belongs to the universality class of the Harper model with both E ($t_b < t_a$) and L ($t_b > t_a$) phases and a critical point at $t_a = t_b$ where the system has the full square symmetry. On the other hand, for $2t_{ab} \geqslant t_a$ the model is found to belong to a new universality class where there is no E phase but instead the C phase exists over a finite parameter interval $t_b \leqslant 2t_{ab}$. For $t_b > 2t_{ab}$ the states are exponentially localized. The E phase and the C phase are separated by a bicritical line $2t_{ab} = t_a$.

Detailed decimation studies [5] show that the wave functions within the fat C phase above the bicritical line are self-similar (at the band edges) only at certain special values of the parameters. These special points correspond to universal limit cycles of the renormalization. However, for generic parameter values, the fractal characteristics of the critical wave functions do not exhibit self-similarity and are conjectured to be described by a strange attractor of the renormalization flow.

The renormalization analysis of the L phase shows that the fluctuations of the wave functions in the L phase mimic the behaviour in the C phase [3]. The L phase of the Harper universality class is described by a renormalization fixed point of the strong-coupling limit $t_b/t_a \to \infty$ while the L phase above the bicritical line is described by a strange attractor of the associated renormalization [3].

4. Anisotropic NNN couplings

Here we first summarize our results for the phase diagram in the case of anisotropic NNN couplings $t_{ab} \neq t_{a\bar{b}}$. This will be followed by various details of our analysis which include the duality transformations [2] as well as the renormalization analysis.

4.1. A summary of the phase diagram

In addition to distinguishing the phases as E, L and C, we also characterize them in terms of their universality classes defined by the decimation scaling ratio ζ, as follows.

(1) As soon as α and β differ, $\alpha - \beta$ is an irrelevant parameter and the phase diagram in terms of the parameters $(\alpha + \beta)/2$ and λ is the same as the one obtained by setting $\beta = 0$ (provided that $\alpha > \beta$).

(2) For $(\alpha+\beta)/2 < 1$, the asymptotic scaling properties of the wave functions are given by the universality classes of the NN square lattice, described by the Harper equation. The wave functions are in general complex but the corresponding scaling factors are real taking the same values as for $(\alpha + \beta)/2 = 0$.

(3) For $(\alpha + \beta)/2 > 1$, the universality classes are the same as for the Harper equation up to a complex phase factor in ζ.

(4) Along the line $(\alpha + \beta)/2 = 1$, the scaling properties belong to the universality class of the triangular lattice (up to a complex phase).

Table 1 lists the universal scaling ratios for the C and L phases in various universality classes. In the case of the square lattice, there exist two universal scalings which characterize the C and L phases. However, in the case of the triangular lattice, there are three sets of universal ratios describing the critical properties along the line $t_a = (t_{ab} + t_{a\bar{b}}) > t_b$, at the point $t_a = t_b = (t_{ab} + t_{a\bar{b}})$ and along the supercritical (localized) line $t_a = (t_{ab} + t_{a\bar{b}}) < t_b$.

Table 1. The absolute value of the universal scaling ratios for various universality classes at the minimum band edge. $\phi_c = 1/2$ for all other parts of the parameter space except along the line AC where ϕ_c varies as a function of λ.

Universality class and phase	ζ
Harper: C phase	0.211
Harper: L phase	0.176
Asymmetric triangular: C phase	0.825, 0.908, 0.836
Symmetric triangular: C phase	0.238, 0.303, 0.291
Triangular: L phase	0.267, 0.311, 0.121

In the following, we take various limits to obtain a better understanding of the numerical observations on the phase diagram.

4.2. The limit $\alpha \to \infty$, $\beta/\alpha \to 0$

In order to explain the observation (3) of the previous section and in particular the re-entrant E phase, it is useful to consider the limit $t_{ab}/t_a \to \infty$, $t_{a\bar{b}}/t_{ab} \to 0$. Rearranging the resulting TBE, it can be written as

$$C_{k+1} + C_{k-1} + \frac{2t_b}{t_{ab}} \cos[2\pi(k\sigma + \phi)]C_k = \frac{E}{t_{ab}} C_k \tag{8}$$

where C_k is related to ψ_k via

$$C_k = \exp(\mathrm{i}\,2\pi\phi k)\exp(\mathrm{i}\pi\sigma k^2)\psi_k. \tag{9}$$

The appearance of such a transformation is not surprising as the limit corresponds to an oblique lattice which is topologically similar to the square lattice.

The presence of the phase factor in equation (9) makes the RG analysis of the re-entrant E phase more complicated than that of the Harper-type E phase. The re-entrant E phase is described by a complex 6-cycle with complex scaling ratios. This is in contrast to the Harper E phase which is described by a real fixed point. Furthermore, the latter depends neither on the phase ϕ nor on the lattice index k and can be easily solved from a fixed-point equation: $f_n(k) \equiv \sigma$ and $e_n(k) \equiv -\sigma^2$ resulting in $\zeta = 1$. In the complex E phase, complications arise from the fact that the decimation functions do depend both on ϕ and k. But the absolute value of a decimation function has the same constant value as in the real case. The above equation shows that the scaling factors ζ_j exist only for special (i.e. rational or those related to the golden mean) values of the phase ϕ. The relation $F_n\sigma = F_{n-1} - (-\sigma)^n$ implies that

$$\sigma F_n^2 = F_{n-1}F_n + \frac{(-1)^{n-1} + \sigma^{2n}}{1+2\sigma}. \tag{10}$$

From this and equation (9) it follows that for $\phi = 0, 1/2$

$$\zeta_j = \pm\exp[\pm\mathrm{i}\pi/(1+2\sigma)]. \tag{11}$$

Taking advantage of the above limiting solution, it is possible to derive an explicit expression for the 6-cycle. Substituting equation (9) into the decimation equation (3), we obtain

$$\begin{aligned} e_n(k) &= e_n^h(k)\exp(-\mathrm{i}\,2\pi\phi F_n)\exp[-\mathrm{i}\pi\sigma(F_n^2 + 2kF_n)] \\ f_n(k) &= f_n^h(k)\exp(\mathrm{i}\,2\pi\phi F_{n-1})\exp[\mathrm{i}\pi\sigma(F_{n+1}^2 - F_n^2 + 2kF_{n-1})] \end{aligned} \tag{12}$$

where e_n^h and f_n^h are the decimation functions corresponding to the Harper equation (8). From these equations we see that e_n and f_n are functions of the fractional part of $k\sigma$, denoted by $\{k\sigma\}$, only. Therefore, we can write the decimation functions in terms of the renormalized variable $x = (-\sigma)^{-n}\{k\sigma\}$ [7]. For simplicity, let us assume that $\phi = 0, 1/2$. Applying the relation (10) and the fact that

$$(-\sigma)^n F_n = \frac{(-1)^n - \sigma^{2n}}{1+2\sigma} \tag{13}$$

we obtain six different limiting function pairs of the form

$$\begin{aligned} e^*(x) &= \pm\sigma^2\exp[\pm\mathrm{i}\pi(2x-1)/(1+2\sigma)] \\ f^*(x) &= \pm\sigma\exp[\pm\mathrm{i}\pi\,2(\sigma x+1)/(1+2\sigma)] \end{aligned} \tag{14}$$

as n tends to infinity. These pairs form a 6-cycle of the recursion (4), (5), written in terms of the continuous variable x [7]:

$$e_{n+1}(x) = -\frac{Ae_n(-\sigma x)}{1+Af_n(-\sigma x)} \tag{15}$$

$$f_{n+1}(x) = \frac{f_{n-1}(\sigma^2 x+\sigma)f_n(-\sigma x-1)}{1+Af_n(-\sigma x)} \tag{16}$$

$$A = e_{n-1}(\sigma^2 x+\sigma) + f_{n-1}(\sigma^2 x+\sigma)e_n(-\sigma x-1).$$

For the above 6-cycle, $1 + Af^*(-\sigma x) \equiv \sigma$.

Therefore, the characteristic feature of the re-entrant E phase is the fact that the decimation functions depend explicitly on x. These functions are complex and consequently the universal scaling ratio has both real and imaginary parts but the absolute value of ζ is unity. This is unlike the Harper E phase where the real universal functions are site independent and are given in terms of the powers of the golden mean.

In fact, in the analysis above, the only thing which referred to the E phase was the replacement of the functions $e_n^h(k)$ and $f_n^h(k)$ by their asymptotic limits $-\sigma^2$ and σ, respectively. Equally well, we could relate the scaling properties in the C and L phase of this limit to those of the Harper equation.

The fact that the RG fixed points of the 'oblique' limit attract the RG flow also in other parts of the region $(\alpha+\beta)/2 > 1$ is a nontrivial numerical observation which we cannot explain analytically.

4.3. The line ACL: $t_a = t_{ab}, t_{a\bar{b}} = 0$

Without loss of generality we set $t_a = t_{ab} = 1$. Rearranging the phase factors, the complex TBE can be written as the following TBE with real coefficients:

$$\frac{E}{2}\bar{\psi}_k = \cos\left[\pi\left(\sigma k + \phi + \frac{\sigma}{2}\right)\right]\bar{\psi}_{k+1} + \cos\left[\pi\left(\sigma k + \phi - \frac{\sigma}{2}\right)\right]\bar{\psi}_{k-1} + \lambda\cos[2\pi(\sigma k + \phi)]\bar{\psi}_k \tag{17}$$

where

$$\bar{\psi}_k = \exp[\mathrm{i}\pi(k^2\sigma/2 + k\phi)]\psi_k. \tag{18}$$

Note that above TBE describes the triangular lattice [10]. We next show that the scaling properties for the triangular lattice are different from those of the square lattice. It turns out that the line AC is critical and, except for the point C on this line, the line is described by a unique fixed point.

We first consider the regime to the left of the point C where $t_a = t_{ab} > t_b$. The RG analysis in this case shows that t_b is an irrelevant parameter and hence the scaling properties along this line are described by a unique fixed point. By setting $t_b = 0$ ($t_a = t_{ab} = 1$) the TBE in this case can be written as

$$(1 + \exp[\mathrm{i}\,2\pi((k+1/2)\sigma + \phi)])\psi_{k+1} + (1 + \exp[-\mathrm{i}\,2\pi((k-1/2)\sigma + \phi)])\psi_{k-1} = E\psi_k. \tag{19}$$

Using the Fourier transformation

$$\psi_k = \sum g_l \exp(\mathrm{i}\,2\pi\sigma l k) \tag{20}$$

the above TBE can be written as

$$\bar{g}_{l+1} + \bar{g}_{l-1} + 2\cos(2\pi\sigma l)\bar{g}_l = E\bar{g}_l \tag{21}$$

where

$$\bar{g}_l = \exp[-\mathrm{i}\pi(\sigma l^2 - 2l\phi)]g_l. \tag{22}$$

This is identical to the Harper equation at its self-dual point. The above transformation not only illustrates the fact that the point A is critical but it also shows that the scaling ratios at that point are linear combinations (weighted over a pseudo-random phase factor) of an infinite set of scaling ratios of the one-dimensional lattice points and hence are different from that of the Harper critical point. However, this analysis shows that the two systems are related in a complicated way and have the same spectrum.

Our RG analysis shows that at the point C where $t_a = t_{ab} = t_b$, the scaling properties are different from those of the rest of the line AC as shown in table 1.

For the case where $t_a = t_{ab} < t_b$, we follow the method described in our previous paper to study the L phase [3]. Again, the scaling ratios along the line CL in the L phase are different from the scaling ratio describing the L phase of the Harper universality class.

5. The relationship with the quantum Ising model

It turns out that the TBE in the triangular-lattice limit also describes another problem, namely the quantum Ising model (QIM) in a transverse field h_k:

$$H = -\sum \sigma_k^x \sigma_{k+1}^x - h_k \sigma_k^z. \tag{23}$$

Using the methods of Lieb *et al* [11], the eigenvalue equation for the spin problem can be written in a TBE form, which for $h_k = \lambda \cos[\pi(k\sigma + \phi)]$ is

$$E\psi_k = \cos[\pi(k\sigma + \sigma + \phi)]\psi_{k+1} + \cos[\pi(k\sigma + \phi)]\psi_{k-1} + \frac{\lambda}{2}\cos[2\pi(k\sigma + \phi)]\psi_k. \tag{24}$$

Here $E = -\lambda/2 + (\bar{E}^2/4 - 1)/\lambda$, where $\bar{E}$ is the energy of the Ising model (23).

It is rather interesting to note the similarities between these two seemingly unrelated problems. Both are described by a NN TBE where the diagonal as well as off-diagonal terms are modulating: the periodicity of the diagonal term is twice the periodicity of the off-diagonal term. The only difference between the models is in the relative σ-dependent phase differences between the diagonal and off-diagonal terms.

The QIM was recently studied for $\sigma = (\sqrt{5}-1)/2$ [12]. For $\lambda \leqslant 2$, the model exhibited critical states which became exponentially localized for $\lambda > 2$. This localization transition was accompanied by the magnetic transition to long-range order driven by a conformally invariant quantum state. Very recent decimation studies [7] confirmed this phase diagram of the model and showed that the subconformal regime was described by a unique fixed point of the renormalization flow. However, at the onset of localization, the renormalization flow was attracted by a different fixed point resulting in a different universality class.

Comparison of the RG analysis for the QIM and the triangular-lattice model along the line ACL shows that in the subconformal and the strong-coupling phases of the Ising model the RG flow approaches asymptotically the same 3-cycle as for the asymmetric and strong-coupling triangular lattices for both the upper and the lower band edges (maximum and minimum energy). At the conformally invariant point of the QIM, corresponding to the onset of localization, the universal characteristics of the triangular-lattice band edges are the same as those of the upper band edge of the QIM. However, the lower band edge of the QIM has zero energy and the corresponding state is believed to be conformally invariant. Unlike the upper-band-edge 3-cycle, the renormalization flow for the decimation functions at $\bar{E} = 0$ converges to a period-1 fixed point. In addition, the wave function at the conformal point is asymmetrical, vanishing on one side of main peak [7]. The ‘conformal’ renormalization fixed point of the QIM does not map to any quantum state of the triangular model. However, it turns out that the conformal fixed point is identical to the strong-coupling fixed point of the Harper equation describing the self-similar fluctuations in the localized states [3]. This interesting result is due to the fact that the TBE describing the conformal state of the Ising model is related to the TBE describing the strong-coupling limit of the Harper equation. Therefore, although the Ising model is described almost everywhere by the triangular universality class, the onset to the long-range order corresponds to the square-lattice universality class.

6. Conclusions

In summary, we have shown that almost all of the four-dimensional phase diagram of the generalized Harper equation can be described by two universality classes, namely the universality class of the square lattice (the Harper equation) and the universality class of the triangular lattice. The only exception is the region where the two NNN couplings are equal and are greater than one half of one of the NN couplings. Furthermore, the triangular universality class also describes the phase diagram of the QIM with the only exception being the conformally invariant quantum state characterizing the magnetic transition to long-range order. The conformal state is described by the strong-coupling limit of the Harper equation.

We have shown that with anisotropic NNN couplings, the renormalization behaviour for the TBE describing an electron on a square lattice is a lot simpler than in the isotropic case. Firstly, the renormalization strange set corresponding to the fat C phase in the isotropic case is replaced by an attracting cycle associated with trivial scaling properties (i.e. the E phase). Secondly, the bicritical lines of the isotropic case remain critical also when the NNN couplings are not equal but the renormalization attractor is again simpler (i.e. a cycle). Thirdly, the fluctuations of the exponentially localized wave functions are described by a universal fixed point and not by an infinite strange set as in the isotropic case. A novel and very interesting result of our studies is the fact that the triangular universality class is sandwiched between two Harper-type phases. Therefore, as one of the parameters is varied, we have the E–E transition, in addition to the E–L and C–L transitions.

Acknowledgments

The research of IIS is supported by a grant from National Science Foundation: DMR 093296. JAK would like to thank the Niilo Helander Foundation for financial support. IIS would like to acknowledge the hospitality of the Magnetic Materials group at the National Institute of Standards and Technology where part of this research was done.

References

[1] Harper P G 1955 *Proc. Phys. Soc.* A **68** 874
[2] For a review, see
Sokoloff J B 1985 *Phys. Rep.* **126** 189
Simon B 1982 *Adv. Appl. Math.* **3** 463
Last Y 1995 *Proc. 11th Int. Congr. on Mathematical Physics* ed D Iagolnitzer (Boston, MA: International Press Inc.) p 366
Jitomirskaya S Ya 1995 *Proc. 11th Int. Congr. on Mathematical Physics* ed D Iagolnitzer (Boston, MA: International Press Inc.) p 373
[3] Ketoja J A and Satija I I 1995 *Phys. Rev. Lett.* **75** 2762
[4] Han J H, Thouless D J, Hiramoto H and Kohmoto M 1994 *Phys. Rev.* B **50** 11 365
Hatsugai Y and Kohmoto M 1990 *Phys. Rev.* B **42** 8282
[5] Ketoja J A, Satija I I and Chaves J C 1995 *Phys. Rev.* B **52** 3026
[6] Ostlund S and Pandit R 1984 *Phys. Rev.* B **29** 1394
Ostlund S, Pandit R, Rand D, Schellnhuber H J and Siggia E D 1983 *Phys. Rev. Lett.* **50** 1873
[7] Ketoja J A and Satija I I 1994 *Phys. Lett.* **194A** 64
Ketoja J A and Satija I I 1995 *Physica* A **219** 212
[8] Ketoja J A 1992 *Phys. Rev. Lett.* **69** 2180
[9] The p-cycle is observed for the phase $1/2$ (or 0) and for the critical phase ϕ_c for which the main peak of the wave function is at $k = 0$ (in many cases $\phi_c = 1/2$). However, for an arbitrary phase the asymptotic behaviour of the decimation functions is more complicated. In the numerical work, we have used this

cycle property to improve the initial estimate for E_{min} obtained by diagonalizing a large matrix (see reference [7]).
[10] Claro F H and Wannier G H 1979 *Phys. Rev.* B **19** 6068
Thouless D J 1983 *Phys. Rev.* B **28** 4272
[11] Lieb E, Schultz T and Mattis D 1961 *Ann. Phys., NY* **16** 407
[12] Satija I I and Doria M 1989 *Phys. Rev.* B **39** 9757